Research Article

2 DOMINATION NUMBER AND 2 BONDAGE NUMBER OF COMPLETE GRID GRAPH
Dr. D. K. Thakkar, D. D. Pandya

Address for Correspondence
Department of Mathematics, Saurashtra University, Gujarat.

ABSTRACT
Grid graphs and domination are very important ideas in computer architecture and communication techniques. We present results about 2 Domination Number and 2 Bondage Number for Grid Graphs. We find 2 dominating sets and 2 domination number and 2 bondage number for $G_{m,n}$ using special patterns.

KEYWORDS: Grid graph, Complete grid graph, Domination number, 2 Domination number, 2 Bondage number.

1. INTRODUCTION
Starting in the eighties domination numbers of Cartesian products were intensively investigated. In the meantime, some papers on domination numbers of cardinal products of graph was initiated by Vizing[10]. He conjectured that the domination number of the Cartesian product of two graph is always greater than or equal to the product of the domination numbers of the two factors. This conjecture is still unproven. For complete grid graphs, i.e. graphs $P_k \times P_n$, algorithms were given for a fixed k which compute $\gamma(P_k \times P_n)$ in O(n) time [7]. In fact, the domination number problem for $k \times n$ grids, where k is fixed, has a constant time solution. In this paper we present a survey of 2 domination numbers and 2 bondage numbers of complete grid graphs $P_k \times P_n$, K=2,3,4,5.

2. DEFINITIONS
For notations and graph theory terminology, we follow Bondy and Murthy [1].
Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. A subset $D$ of $V$ is a dominating set of $G$ if every vertex of $V \setminus D$ is adjacent to at least one vertex of $D$. The minimum cardinality of a dominating set is called the domination number of $G$ which is denoted as $\gamma(G)$.

A set $D$ of vertices of a graph $G$ is a 2 dominating set if every vertex of $V \setminus D$ is adjacent to at least two vertices of $D$. A 2 dominating set of minimum cardinality is a minimum 2 dominating set and its cardinality is 2 domination number of $G$ denoted by $\gamma_2(G)$.

The 2 bondage number is minimum number of edges whose removal from original graph increase 2 domination number in resultant graph denoted by $\beta_2(G)$.

COMPLETE GRID GRAPH $G_{3,4}$
A two-dimensional complete grid graph is an $m \times n$ graph $G_{m,n} = P_m \times P_n$, the product of path graphs on m and n vertices. The Cartesian products of paths $P_m$ and $P_n$ with disjoint sets of vertices $V_m$ and $V_n$ and edge sets $E_m$ and $E_n$ is the graph with vertex set $V(P_m \times P_n)$ and edge set $E(P_m \times P_n)$ such that $((g_1,h_1),(g_2,h_2)) \in E(P_m \times P_n)$, if and only if either $g_1=g_2$ and $(h_1,h_2)\in E(P_n)$ or $h_1=h_2$ and $(g_1,g_2)\in E(P_m)$. One example of a complete grid graph $G_{3,4} = P_3 \times P_4$ is shown in figure 1.

A grid graph $G_{m,n}$ has mn nodes and $(m-1)n+(n-1)m=2mn-m-n$ edges. We observe that the path graph $P_n = G_{1,n} = G_{n,1}$ and cycle graph $C_4 = G_{2,2}$. From the definition of complete grid graph $P_k \times P_n$ we observe that for k=1 the grid graph is nothing but path graph that is $P_1 \times P_n = P_n \times P_1 = P_n$.

H-MERGE AND V-MERGE OPERATIONS:
From the definition of complete grid graph $P_k \times P_n$ we observe that for k=n=2 the grid graph $P_1 \times P_2$ is a cycle on 4 vertices.

H-Merging of two cycles
Let cycle $C_1$ and $C_2$ be two cycles on 4 vertices. Let cycle $C_1$ have vertex set $\{u_1,u_2,u_3,u_4\}$ and cycle $C_2$ have vertex set $\{v_1,v_2,v_3,v_4\}$ then vertex set of new graph obtained by H-merging denoted as $C_1 \cap C_2$ is
\[ \{u_1, u_2 = v_1, u_3, u_4 = v_3, v_2, v_4\} \]. Edges in \( C_1 \cap C_2 \) includes all the edges of \( C_1 \) and \( C_2 \) with 
\[ (u_2, u_4) = (v_1, v_3) \]
This gives that 
\[ |V(C_1 \cap C_2)| + |V(C_2)| - 2 \]
\[ |E(C_1 \cap C_2)| + |E(C_1)| + |E(C_2)| - 1 \]
Thus H-Merging of two cycles gives a complete grid graph 
\[ G_{2,2} = P_2 \times P_3 = G_{2,3} \]

**V-Merging of two cycles**
In similar way we define another operation V-Merging which gives a complete grid graph shown as below.

**Lemma 1**
The 2-domination number of grid graph 
\[ G_{2,n} = P_2 \times P_n \] for \( n \geq 2 \) is 
\[ \Upsilon_2(G_{2,n}) = n \]

**Proof**
Step : 1 Result is true for \( n = 2 \) 
\[ \Upsilon_2(G_{2,2}) = 2 \]
Step: 2 Suppose result is true for \( n = k \) 
So, \[ \Upsilon_2(G_{2,k}) = k \]
Step : 3 If \( n = k + 1 \) 
To prove: \[ \Upsilon_2(G_{2,k+1}) = k + 1 \]
Consider \( G_{2,k} \) Let D be 2 dominating set in \( G_{2,k} \).
then by figure given below we can say that there will be one vertex from pair which are non adjacent will be in D.

To construct \( G_{2,k+1} \). We have to add 2 vertices in \( G_{2,k} \).

By step -2 \( \Upsilon_2(G_{2,k}) = k \). Now to 2 dominate last two vertices we will have to add one vertex in D. and we will have to add one vertex from pair which is not adjacent to a vertex of D shown below.

SO, \[ \Upsilon_2(G_{2,k+1}) = k + 1 \]

**Lemma 2**
The 2 domination number of complete grid graph 
\[ G_{3,n} = P_3 \times P_n \] for \( n \geq 3 \) is 
\[ \Upsilon_2(P_3 \times P_n) = \begin{cases} 4n/3, & \text{if } n = 3k \\ 4n + 2/3, & \text{if } n = 3k + 1 \\ 4n + 1/3, & \text{if } n = 3k + 2 \end{cases} \]

**Proof:**
Consider grid graph \( P_3 \times P_n \) for \( n \geq 3 \)
For \( n = 3 \), we have \( G_{3,3} = P_3 \times P_3 \). Let D be a 2 dominating Set of \( G_{3,3} \). For this graph we have four corner vertices with degree 2, four boundary vertices of degree 3 & one center vertex of degree 4. To 2 dominate four corner vertices we have three possibilities.

\(<A>\) all four vertices in D
a) Then to 2 dominate these vertices we can include 2 more vertices in D. So D has 6 vertices.
b) Then to 2 dominate these vertices we can include center vertex in D so D has 5 Vertices.

\(<B>\) None of corner vertices into D
Then to 2 dominate, the corner vertices we must include both of its neighbors into D. We can see that none of these are adjacent to each other But the central vertex is adjacent to all these four vertices included into D.
D has four Vertices.
From all the possible cases considered above we get the cardinaly & minimum 2 dominating Set of \( G_{3,3} \) is

\[ 4. \ U_2(G_{3,3}) = 5 \]

**Case-1**
If \( n=3k \).Now divide the \( 3 \times n \) grid graphs into K blocks \( P_3 \times P_3 \) each. Extending the above result for 
\[ P_3 \times P_{3k} \] where each of the K blocks \( P_3 \times P_3 \) contribute 4 vertices into the minimal 2 dominating set.
we get 4K vertices into minimal 2 dominating set.
Using the minimality of 2 dominating set of \(G_{3,3}\), we claim the minimality of 2 dominating set of \(P_3 \times P_{3k}\). Hence \(\gamma_2(P_3 \times P_{3k}) = 4k\).

**Case-2**

If \(n = 3k+1\), we divide the \(3 \times n\) grid graph into \(k\) blocks of \(P_3 \times P_3\) each and a path \(P_3 \times P_1\). As in case-1, we get patterns in the \(k\) blocks of giving 4 vertices into the 2 dominating set from each block. For 2 domination of the vertices in the last block \(P_3 \times P_1\) we need 2 vertices to be added to the dominating set. Thus, the 2 dominating set of \(P_3 \times P_{3k+1}\) has 4\(k+2\) vertices. As in case-1, we claim that this is the minimal 2 dominating set of \(P_3 \times P_{3k+1}\).

\[\gamma_2(P_3 \times P_{3k+1}) = \gamma_2(P_3 \times P_{3k}) + \gamma_2(P_3) = 4k + 2\]

**Case-3**

If \(n = 3k+2\), we divide the \(3 \times n\) grid graphs into \(k\) blocks of \(P_3 \times P_3\) each and a path \(P_3 \times P_2\). As in case-1, we get patterns in the \(k\) blocks of \(P_3 \times P_3\) giving 4 vertices into the 2 dominating set from each block. For 2 domination of the vertices in the last block \(P_3 \times P_2\) we need 3 vertices to be added to the 2 dominating set. Thus, the minimal 2 dominating set of \(P_3 \times P_{3k+2}\) has \(4k+3\) vertices. Hence

\[\gamma_2(P_3 \times P_{3k+2}) = \gamma_2(P_3 \times P_{3k}) + \gamma_2(P_3 \times P_2) = 4k + 3\]

One example of a 2 dominating set for \(P_3 \times P_1\).

\[\gamma_2(P_3 \times P_n) = 4k, \quad \text{if} \quad n = 3k\]

\[= 4k + 2, \quad \text{if} \quad n = 3k + 1\]

\[= 4k + 3, \quad \text{if} \quad n = 3k + 2\]

This can be written as

\[\gamma_2(P_3 \times P_n) = \frac{4n}{3}, \quad \text{if} \quad n = 3k\]

\[= 4n + 2, \quad \text{if} \quad n = 3k + 1\]

\[= 4n + 3, \quad \text{if} \quad n = 3k + 2\]

**Lemma 3:** The 2 domination number of grid graphs \(P_3 \times P_n\) for \(n \geq 2\) is 2\(n\)

**Proof:**

**Step-1** Result is true for \(n = 2\)

\[\gamma_2(P_3 \times P_2) = 4\]

**Step-2** Suppose result is true for \(n = k\)

\[\gamma_2(P_3 \times P_k) = 2k\]

**Step-3** Check result for \(n = k + 1\)

To prove: \(\gamma_2(P_3 \times P_{k+1}) = 2(k + 1)\)

By step 2, \(\gamma_2(P_3 \times P_k) = 2k\)

Let \(D\) be 2 dominating set in \(P_3 \times P_k\) with cardinality \(K\). Now consider \(P_3 \times P_{k+1}\). So, \(\gamma_2(P_3 \times P_{k+1}) = \gamma_2(P_3 \times P_k) + \gamma_2(P_3) = 2k + 2 = 2(k + 1)\)

**Lemma 4:** The 2 domination number of grid graphs \(P_5 \times P_n\) for \(n \geq 2\) is

\[\frac{5n}{2}, \quad \text{if} \quad n = 2k\]

\[\frac{5n - 1}{2}, \quad \text{if} \quad n = 2k + 1\]

**Proof:**

Consider grid graph \(P_5 \times P_n\) for \(n \geq 2\)

From Lemma 5, we get that

\[\gamma_2(P_5 \times P_2) = \gamma_2(P_5 \times P_3) = 5\]

From Lemma 6, we get that

\[\gamma_2(P_5 \times P_3) = \gamma_2(P_5 \times P_4) = 7\]

From Lemma 7, we get that

\[\gamma_2(P_5 \times P_4) = \gamma_2(P_5 \times P_5) = 10\]

Consider \(P_5 \times P_2\).

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Case 1: if \( n = 2k \)

Now divide the \( 5 \times n \) grid graphs into \( k \) blocks \( P_5 \times P_2 \) each. Extending the above result for \( P_5 \times P_{2k} \) where each of the \( k \) blocks \( P_5 \times P_2 \) contribute 5 vertices into the minimal 2 dominating set. We get 5k vertices into minimal 2 dominating set. Using the minimality of 2 dominating set of \( G_{5,2} \), we claim the minimality of 2 dominating set of \( P_5 \times P_{2k} \).

Hence \( \gamma_2(P_5 \times P_{2k}) = 5k \)

Case 2: if \( n = 2k + 1 \)

We divide the \( 5 \times n \) grid graphs into \( k \) blocks of \( P_5 \times P_2 \) each and a path \( P_5 \times P_1 \). As in case 1 we get patterns in the \( k \) blocks of \( P_5 \times P_2 \) giving 5 vertices into the 2 dominating set from each block for 2 domination of the vertices in the block \( P_5 \times P_1 \). We need 2 vertices to be added to 2 dominating set. Thus the 2 dominating set of \( P_5 \times P_{2k+1} \) has 5k+2 vertices.

As in case 1 we claim that this is the minimal 2 dominating set of \( P_5 \times P_{2k+1} \).

Hence

\[ \gamma_2(P_5 \times P_{2k+1}) = \gamma_2(P_5 \times P_{2k}) + \gamma_2(P_2) = 5k + 2 \]

**Lemma 5:** The 2 bondage number of complete grid graph \( G_{2,n} = P_5 \times P_n \) for \( n \geq 2 \) is 1.

**Proof:**

Consider the complete grid graph \( P_5 \times P_n \). By lemma

\[ \gamma_2(P_5 \times P_n) = n \text{ for } n \geq 2 \]

Let \( D \) be a dominating set of \( P_5 \times P_n \). So we have \( n \) alternating vertices of pair of \( G \) in \( D \).

Consider corner vertices which are not in \( D \) those are adjacent to only two vertices of \( D \) whereas remaining are adjacent to three vertices of \( D \).

If we remove one edge adjacent to corner vertices which is not in \( D \) then this particular vertex is now adjacent to only one vertex of \( D \).

So now to 2 dominate this graph we will have to add this vertex into \( D \).

So \( \gamma_2(P_5 \times P_n) \leq \gamma_2(P_5 \times P_n \setminus \{e\}) \)

So \( B_2(P_5 \times P_n) = 1 \)

**Lemma 6:**

The 2 bondage number of complete grid graph \( G_{5,n} = P_5 \times P_n \)

\[ B_2(P_5 \times P_n) = \begin{cases} 1 & \text{if } n = 3k+1, \ k \geq 1 \\ \text{otherwise} & \end{cases} \]

**Proof:**

Case – 1

\[ n \neq 3k + 1, \ k \geq 1 \quad \text{so}, \ n = 3k \text{ or } n = 3k+2, \ k \geq 1 \]

By lemma 6,

\[ B_2(P_5 \times P_n) = \begin{cases} 4n/3 & \text{if } n = 3k \\ 4n+2/3 & \text{if } n = 3k+1 \\ 4n+1/3 & \text{if } n = 3k + 2 \end{cases} \]

Consider \( P_3 \times P_5 \) & \( P_3 \times P_6 \).

Let \( D \) be a dominating set. We can observe that each corner vertices is not in \( D \).

If we remove an edge adjacent to this corner vertex does not belong to \( D \). Then \( D \) is not 2 dominating set for resultant graph. To 2 dominate resultant graph we have to add this vertex in \( D \).

So \( B_2(P_5 \times P_n) = 1 \)

Case -2

\[ n = 3k+1, \ k \geq 1 \]

consider \( P_3 \times P_n \). Let’s see an example of \( P_3 \times P_4 \)

Let \( D \) be a dominating set for \( P_3 \times P_n \). We can observe that among 4 corner vertices there are two vertices are in \( D \). If we remove two edges of vertex which is adjacent to corner vertex belongs to \( D \), then

\[ \gamma_2(P_3 \times P_n) \leq \gamma_2[(P_3 \times P_n) \setminus \{2e\}] \]
So, $\beta_2(P_3 \times P_3) = 2$

**Lemma 7** The 2 bondage number of complete grid graph $G_{4,n} = P_4 \times P_n$ is 2 if $n \neq 1,2$

**Proof:**
Consider the complete grid graph $G_{4,n} = P_4 \times P_n$

By lemma 7, $\gamma_2(P_4 \times P_n) = 2n$ means each column consists of 2 vertices.

For example $P_4 \times P_4$

Let D be 2 dominating set of $P_3 \times P_3$. Consider corner vertices. Among 4 vertices, there are 2 vertices belongs to D.

If we remove an edge having one end vertex belongs to D and another doesn’t belong to D then there will be rearrangement of D such that $\gamma_2(P_3 \times P_3) = \gamma_2[(P_3 \times P_3) \backslash \{e\}]$

If we remove two edges adjacent to vertex doesn’t belong to D and which is adjacent to corner vertex belongs to D.

then $\gamma_2(P_3 \times P_3) \leq \gamma_2[(P_3 \times P_3) \backslash \{2e\}]$

So, $\beta_2(P_3 \times P_3) = 2$

**Case 2: N=odd**
Let D be 2 dominating set of $P_5 \times P_n$.

Consider corner vertices which are not in D those are adjacent to only two vertices of D whereas remaining are adjacent to three vertices of D.

If we remove one edge adjacent to corner vertices which is not in D then this particular vertex is now adjacent to only one vertex of D.

So, now to 2 dominate this graph we will have to add this vertex into D.

So $\gamma_2(P_5 \times P_3) \leq \gamma_2(P_5 \times P_n \backslash \{e\})$

So, $\beta_2(P_5 \times P_n) = 1$

**CONCLUSION**
Jacobson and Kinch [7] have given dominating sets and domination number of grid graphs $P_k \times P_n$ for $k=1,2,3,4$ and for all $n \geq 1$. The domination number of grid graphs $P_k \times P_n$ is denoted by $\gamma(P_k \times P_n)$. Later Hare [16] developed an algorithm to compute the domination number for grid graphs $P_k \times P_n$ and found simple formulas for its computation for $1 \leq k \leq 10$. Chang and Clark [2] proved Hare’s formulas for $k=5$ and 6 and $n \geq 1$. Here we present a list of domination numbers, 2 domination numbers and 2 bondage numbers for grid graphs $P_k \times P_n$ for $k=1,2,3,4,5$.

**REFERENCES**
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3. Deo Narsingh: Graph Theory with Applications to Engineering and Computer Science, Prentice Hall, (2001)
Graph Domination Number 2 Domination Number 2 Bondage Number

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<th>Domination Number</th>
<th>2 Domination Number</th>
<th>2 Bondage Number</th>
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<td>$\gamma_2(P_1 \times P_n)$</td>
<td>$\beta_2(P_2 \times P_n)$</td>
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<td>$n \geq 2$</td>
<td>$1$, $n \geq 2$</td>
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<td>$k \geq 1$, $n = 3k + 1$</td>
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<td>$2n$ for $n \geq 2$</td>
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<td>$\frac{5n}{2}$, $n = 2k$, $n \geq 2$</td>
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