Research Paper

SUITABILITY OF B-SPLINE COLLOCATION TECHNIQUE TO ARRIVE AT APPROXIMATE SOLUTIONS FOR STRUCTURAL PROBLEMS

Deepak Mahapatra
Address for Correspondence
Research Scholar, Department of Mechanical Engineering NIT Raipur, India

ABSTRACT
This work is primarily focused on identifying the potential of B-spline collocation technique in finding approximate solutions for structure based boundary value problems. A mathematical model of an available structure problem (fourth order differential equation for a cantilever beam with given boundary conditions) is considered. A numerical solution to this problem is approached using B-spline collocation technique. A collocation technique involves satisfying the differential equation at finite number of points called collocation points. Here we use the Greville abscissa method to find the collocation points. B-spline basis functions of sixth order are used to approximate the function. Results are then compared with those in the available literature. It is concluded that this technique can be efficient and simpler to find solutions to boundary value problems of structural problems.

INTRODUCTION
Several numerical techniques are available to find approximate solutions to differential equations of mathematical models of engineering problems. One of the techniques available is Collocation technique. A collocation method involves satisfying a differential equation to some tolerance at a finite number of points, called collocation points. A collocation method is based on evaluating the accuracy of a differential equation at a finite set of collocation points. The main advantage of this technique is its simplicity and hence low cost of computation. However the level of accuracy as provided by this technique is not high which limits its applicability to a relatively narrow area.

In the last few years collocation technique using B-spline basis functions is increasingly used to solve engineering problems. The idea is to connect the superior accuracy and smoothness of B-spline basis functions with the low computational cost of collocation. It has been shown that this technique can be efficient and at par to other established techniques of finite element analysis and finite difference technique. The B-spline Collocation Method has a few distinct advantages over the Finite Element and Finite Difference Methods. The advantage over the Finite Difference Method is that the B-spline Collocation Method provides a piecewise-continuous, closed form solution. Finite element formulation has its advantage while working on complex shapes and profiles. An advantage of B-spline collocation methods over the Finite Element Method is that the former procedure is simpler and easy to apply to many problems involving differential equations.

In this paper our aim is to explore the implementation of collocation method using B-spline basis function to solve initial and boundary value problems. We have used the collocation method with B-spline basis functions of third, fourth and fifth degree to find numerical solutions of some linear and non-linear equations. A mathematical model of an available structure problem (fourth order differential equation for a cantilever beam with given boundary conditions) is considered here. A numerical solution to this problem is approached using B-spline collocation technique.

LITERATURE REVIEW
The rapid development of spline functions is primarily due to their great usefulness in applications. Classes of spline functions possess many nice structural properties as well as excellent approximation powers. Splines have many applications in the numerical solution of a variety of problems in applied mathematics and engineering. Splines are types of curves, originally developed for shipbuilding in the days before computer modeling. B-splines have been developed primarily to aid the CAD/CAM people in dealing with complex geometries [3]. It is commonly accepted that the first mathematical reference to it was made in the year 1946 by Schoenberg, however actual thrust for splines came in 1970s after the work of deBoor [8, 9] who provided the subroutine package for drawing and using the b-spline curves.

Excellent survey of the b-spline collocation techniques is presented in [1, 2] with valuable references. Botella [11] used a nodal technique that took the maximum of the B-spline basis functions as collocation points for solving the Navier-Stokes equation, but he could not account for the pressure terms. DeBoor and Swartz [9] used an orthogonal collocation approach that used the Gaussian points as collocation points. The Gaussian points are the zeroes of the appropriate degree Legendre polynomials over the normalized knot interval. Johnson [6] compared using nodal and orthogonal collocation with nodal collocation at the Greville Abscissa points.

After a thorough review it was found that several authors have published their works on the application of B-spline Collocation Method in the solution of boundary value problems (BVP) in the form of differential equations. Some of the cases studied including the one-dimensional heat equation [7], the boundary value problems for a system of singularly perturbed second order ordinary differential equations [5], and advection-diffusion equations [7], solution of radiation-conduction problem [12]. It is also observed that this technique has been effectively and efficiently used in the field of heat transfer and fluid mechanics problems. Its application in structural problems is comparatively new and it is felt intuitationally that this technique will be equally beneficial in this area as well. In [13] Magoon has applied the technique to solve a geometrically non-linear beam problem.

Formulation
In this method a function is approximated by passing a polynomial through values of the function at selected points. The selected points are known as
collocation points. The method begins with a proper choice of basic functions \( \{\varphi_0, \varphi_1, \ldots, \varphi_n\} \) and a set of points \( a = x_1 < x_2 < \ldots < x_n = b \) called nodes or collocation points in the domain \([a, b]\).

The approximate solution can be written in the form

\[
u = \sum_{i=1}^{n+1} \xi_i \varphi_i \]

This method requires that this approximation must satisfy the given differential equation at each of the nodes and also satisfy the boundary conditions. Here ‘\( n \)’ can be thought of as the quality parameter, as \( n \) increase, the error in the approximation must reduce. The choice of \( \varphi \) can vary depending on the problem.

A spline is a continuous piecewise curve used to approximate a solution to a mathematical problem. The piecewise polynomial approximation allows us to construct highly accurate approximations. A spline is continuously differentiable up to a limit defined by its order/degree. Hence it is continuously smooth curve. A spline curve is dependent upon a relationship between the basis function and the vertices of a defining polygon.

Using \((n+1)\) control points \( P_0, P_1, \ldots, P_n \) Basis-Spline (or B-spline) defined as-

\[
P(t) = \sum_{i=1}^{n+1} N_{ik}(t)P_i \]

B-spline basis functions \( (N_i(t)) \) are defined using the Cox-deBoor recursive formula as-

\[
N_{ik}(t) = \begin{cases} 1, & x_i \leq t < x_{i+1} \\ 0, & \text{otherwise} \end{cases} 
\]

\[
N_{ik}(t) = \frac{(t-t_i)N_{i,k-1}(t)}{x_{i+k}-x_i} + \frac{(x_{i+k+1}-t)N_{i+1,k-1}(t)}{x_{i+k+1}-x_{i+1}} \]

where \( 'k' \) is the order of the B-spline curve and knot vector \( T = [x_1, x_2, x_3, \ldots, x_{n+1}] \)

\[
N_i(t) = \begin{cases} 1, & x_i \leq t < x_{i+1} \\ 0, & \text{otherwise} \end{cases} 
\]

\[
N_{ik}(t) = \begin{cases} (t-x_i)N_{ik-1}(t), & x_i \leq t < x_{i+1} \\ N_{ik-1}(t), & \text{otherwise} \end{cases} 
\]

\[
N_{ik}(t) = \begin{cases} (t-x_i)N_{ik-1}(t), & x_i \leq t < x_{i+1} \\ N_{ik-1}(t), & \text{otherwise} \end{cases} 
\]

\[
N_{ik}(t) = \begin{cases} (t-x_i)N_{ik-1}(t), & x_i \leq t < x_{i+1} \\ N_{ik-1}(t), & \text{otherwise} \end{cases} 
\]

\[
N_{ik}(t) = \begin{cases} (t-x_i)N_{ik-1}(t), & x_i \leq t < x_{i+1} \\ N_{ik-1}(t), & \text{otherwise} \end{cases} 
\]

Solution

The Greville abscissa is calculated using the formula mentioned above as-

\[
x_1 = 0, x_2 = 1/5, x_3 = 2/5, x_4 = 3/5, x_5 = 4/5, x_6 = 1
\]

The trial function \( u = \sum_{i=1}^{n+1} u_i \) i.e.,

\[
P(t) = B_1(1-t)^3 + B_2[5t(1-t)^2] + B_3[10t^2(1-t)] + B_4[5t^3] + B_5[5t^2(1-t)] + B_6[t^3]
\]

where B’s are the control point coordinates or we can say weights specific to each basis function.

As we have four equations corresponding to the boundary conditions we need two collocation points to obtain the remaining equations. The intermediate points \( x_3 = 2/5, x_4 = 3/5 \) are selected as collocation points. The obtained equations are simultaneous linear equations that can be solved easily to obtain the values of unknown B’s as follows-

\[
B_1 = 0, B_2 = 0, B_3 = -0.0144, B_4 = -0.0421, B_5 = -0.0824, B_6 = -0.1343,
\]

The exact solution found in literature is-

\[
P(x) = \frac{x^2}{24EI} [4Lx - x^3 - 6L^2]
\]

Results

Relative percentage error is then calculated between the exact and the obtained solution at the collocation points found using Greville abscissa technique as above and the values are tabulated in Table 1.

<table>
<thead>
<tr>
<th>Collocation Point Evaluation</th>
<th>Approximate</th>
<th>Exact</th>
<th>%Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>-5.67E-08</td>
<td>-5.67E-08</td>
<td>3.3E-09</td>
</tr>
<tr>
<td>0.4</td>
<td>-2.24E-07</td>
<td>-2.24E-07</td>
<td>0.0E-09</td>
</tr>
<tr>
<td>0.6</td>
<td>-4.97E-06</td>
<td>-4.97E-06</td>
<td>0.0E-09</td>
</tr>
<tr>
<td>0.8</td>
<td>-5.72E-06</td>
<td>-5.72E-06</td>
<td>0.0E-09</td>
</tr>
<tr>
<td>1</td>
<td>-1.34E-05</td>
<td>-1.34E-05</td>
<td>0.0E-09</td>
</tr>
</tbody>
</table>

Table 1 (Error at collocation points)
Error analysis using different order of basis functions with and without the use of intermediate points is then carried out to determine the comparative applicability of basis functions. It is seen that increasing the order increases the accuracy. Increasing the number of intermediate points also increase the accuracy. The computational time and cost in increasing the order is more as compared to the latter.

CONCLUSIONS
A numerical study of the applicability of B-spline collocation technique in structural problems has been carried out. A fourth order differential equation for a cantilever beam with given boundary conditions is solved using this technique and error is tabulated and plotted at various collocation points as given by Greville abscissa method and compared with a standard solution. Error analysis for various order of basis functions with and without the use of intermediate points is then carried out and tabulated to determine the comparative applicability of basis functions.

REFERENCES
12. Chawla T C, Chan H S “Solution of radiation-conduction problem with collocation method using B-