LIST PRECOLORING EXTENSIONS ON PLANAR GRAPHS

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ABSTRACT

Graph coloring is a well-known and well-studied area of graph theory with many applications. In this paper, we will discuss list precoloring extensions.

KEY WORDS: 5-list-coloring, 2-connected, P -separating 3-cycle.

1.1 INTRODUCTION

In this chapter we will explore the following question posed by Albertson :

**Question 1.1.** Let G be a plane graph. Is there a d > 0 such that whenever P ⊆ V is such that dist(P) ≥ d, then every precoloring of P extends to a 5-list-coloring of G?

Tuza and Voigt [60], see also , showed that the condition of a large distance between precolored vertices is essential by finding a planar graph G with a set of precolored vertices P with dist(P) ≥ 4 such that the precoloring is not extendable to a 5-list-coloring of G. So, the distance d in the above question should be at least 5. Does this question have a positive answer if d ≥ 1000? The original theorem of Thomassen implies that if there are two adjacent precolored vertices assigned distinct colors, then the precoloring is extendable to a 5-list-coloring of G. B’ohme, Mohar, and Stiebitz described when the precoloring of a vertex on a short face with at most six vertices can be extended to a 5-list-coloring of a planar graph.

We introduce a technique using shortest paths in planar graphs which allows us to answer Albertson’s question for a wide class of planar graphs. We prove that a proper precoloring of a pair of vertices can always be extended to a 5-list-coloring of a planar graph provided they are not separated by 3- or 4-cycles. We also provide results about extensions of precolorings of vertices on one face. Finally, we answer Albertson’s question in the case where there are no 3- or 4-cycles separating precolored vertices and there is a special tree containing all of the precolored vertices.

To state our main results in all their generality, we need to define some additional notions. For a path S, with endpoints u and v, we say a vertex w is central if the distances in S from w to u and from w to v differ by at most 1. Note there are at most two central vertices in S. For graph theoretic terminology not defined here, we refer the reader to . By known Lemma , we assume all graphs in consideration are 2-connected. Where necessary, if the application of this lemma causes the distance between a pair of vertices to decrease, we assume the graph the result is applied to is the graph obtained by applying known Lemma. This is important for the results that have a lower bound on a distance constraint.

**Definition 1.2.** Let G be a planar graph, P a subset of vertices of G. Fix a positive integer d. Let T be a tree with P ⊆ V (T). Let the set of special vertices be the union of P and the set of vertices of degree either 1 or at least 3 in T. A path in T with special vertices as endpoints and containing no other special vertices is called a branch of T. We say a tree T is (P, d)-Steiner if

1. every branch has length at least 2d,
2. every branch is a shortest (in G) path between its endpoints,
3. if v is a center of a branch of T, then a shortest (in G) path between v and every vertex in another branch has length at least d, and
4. no two vertices of T from distinct branches have a common neighbor outside of T nor are they adjacent.

For example, when P = {u, v} is a set of two vertices at distance 30 from each other, a shortest (u, v)-path is a (P, 15)-Steiner tree with a single branch.

Next, we define some terms that will be used to simplify the statement of one of the main theorems, as well as a previous result of B’ohme et al. [14].

**Definition 1.3.** Let G = (V, E) be a plane graph and let C be the cycle that corresponds to the boundary of a face of G. Let P = {x0, x1, ..., xk} ⊆ C , where the vertices of P are labeled cyclically around C .

1. The vertex u ∈ V - P is called a bad vertex if u is adjacent to at least five vertices of C .
2. The edge u0 u1 ∈ E, u0, u1, V(G) - P , is called a bad edge if k = 6 and u0, u1, u3, u5 ∈ {x3i+1, x3i+2, x3i+3, x3i+4} for i = 0, 1, where addition of indices is modulo 6.
3. The triangle (u0, u1, u2) , u0, u1, u2 ∈ V - P , is called a bad triangle if k = 6 and the vertex ui, i = 0, 1, 2, where addition of indices is modulo 6.

If a vertex is a bad vertex or part of a bad edge or a bad triangle, it is called an exceptional vertex.

We now state the main results of this chapter.

![Figure 3.1 Exceptional vertex u1](image)

**Theorem 1.4.** Let G be a plane graph, let P be a set of vertices such that there is no P -separating 3-cycle or 4-cycle in G. If there is a Type I reduced graph of G that has a(P,45)- Steiner tree, then every precoloring of P is extendable to a proper 5-list-coloring of G.

**Theorem 1.5.** Let G be a plane graph and u,v ∈ V (G). If G has no {u,v}-separating 3-cycle or 4-cycle, then every proper precoloring of {u, v} is extendable to a proper 5-list-coloring of G.

**Theorem 1.6.** Let G = (V,E) be a plane graph and let...
C be the cycle that corresponds to the boundary of a face of G. Let P = \{v_0, v_1, ..., v_k\} \subseteq V(C), where the vertices of P are labeled cyclically around C. Then every proper precoloring of P is extendable to a 5-list-coloring of G if one of the following conditions holds:
1. G[P] consists of disjoint vertices and edges with pairwise distance at least 3,
2. k ≤ 6 and none of the following occurs:
   a. G contains a bad vertex u \in V - P and L(u) consists of exactly five of the colors assigned to five of the neighbors of u in P,
   b. k = 6, G contains a bad edge u_1, u_2 and there is a color a such that, for i = 1, 2, L(u_i) consists of a and the colors assigned to the four neighbors of u_i in P,
   c. k = 6, G contains a bad triangle (u_1, u_2, u_3) and there are colors a, b, c such that, for i = 1, 2, 3, L(u_i) consists of a, b and the colors assigned to the three neighbors of u_i in P.

**Theorem 1.7.** Let P be a set of vertices in a plane graph G, \text{dist}(P) ≥ 3, such that there are two faces F_1, F_2 where the vertices of P lie on the boundaries of F_1 and F_2. Assume G contains no P-separating 3-cycle or separating 4-cycle. Then every precoloring of P is extendable to a proper 5-list-coloring of G.

The rest of the chapter is organized as follows. In Section 3.2, we describe the origin of Albertson’s question and related results, state known results mentioned above in detail, and prove some technical lemmas. We prove all of the theorems in Section 3.3. Finally, we state open problems and comments in Section 3.4.

**1.2 PRELIMINARIES**

As mentioned earlier, Albertson [2] was able to answer Thomassen’s question about precoloring extensions on planar graphs.

**Theorem 1.8 (Albertson).** Let G = (V, E) be a planar graph and P \subseteq V such that \text{dist}(P) ≥ 4. Then any 5-coloring of P can be extended to a 5-coloring of G.

**Proof.** Let c be a 5-coloring of P using colors \{α, β, γ, δ, γ\} and c be an arbitrary 4-coloring of G using colors \{α, β, γ, δ\}. The goal is to modify c so that it agrees with c. If there is a vertex v \in P for which c(v) \neq c(v), redefine c so that v were assigned color c(v) so that they are now assigned color. Now redefine c so that c(v) = c(v). Since dist(P) = 4, there are no two adjacent vertices of G are both colored. Thus, c is a proper 5-coloring of G extended from a 5-coloring of P. If 5-coloring is replaced with 6-coloring in Thomassen’s question, then it works for dist(P) ≥ 3.

**Theorem 1.9 (Albertson).** Let G = (V, E) be a planar graph and P \subseteq V such that dist(P) ≥ 3. Then any 6-coloring of P can be extended to a 6-coloring of G.

**Proof.** Let c be an arbitrary 6-coloring of P and let G := G-P. Let L be an assignment of lists of size 5 to the vertices of G, all of which are subsets of the list \{α, β, γ, δ\}. Additionally, for a vertex v in G, make sure that L(v) does not contain a color assigned to a vertex of P that is adjacent to v in G. Since dist(P) ≥ 3, each vertex of G is adjacent to at most one vertex of P, so there are at least 5 colors available for L(v).

By Thomassen’s 5-list-coloring theorem, G is L-colorable and this provides a 6-coloring of G extended from a 6-coloring of P.

Note that in the proofs of the previous two results, the planarity of G is only used in the fact that planar graphs are 4-colorable and 5-list-colorable. Thus, the previous two results can easily be generalized to the following which do not require planarity of the graphs in consideration:

**Theorem 1.10 (Albertson).** Let G = (V, E) be a graph that is k-colorable and let P \subseteq V such that \text{dist}(P) ≥ 4. Then any (k + 1)-coloring of P is extendable to a (k + 1)-coloring of G.

**Theorem 1.11 (Albertson).** Let G = (V, E) be a graph that is k-list-colorable and let P \subseteq V such that \text{dist}(P) ≥ 3. Then any (k + 1)-coloring of P is extendable to a (k + 1)-coloring of G.

As mentioned previously, if there is a positive answer to Albertson’s question, then \text{dist}(P) > 4.

**Theorem 1.12 (Tuzar & Voigt).** There is a planar graph G = (V, E) and a set P \subseteq V of vertices such that \text{dist}(P) ≥ 4 and a list assignment L : V → 2^k such that |L(v)| = 3 for all v \in P and |L(v)| = 5 for all v \in V - P such that G is not L-colorable.

As mentioned previously, Böhm et al. described when a precoloring of vertices on a small face is extendable.

**Theorem 1.13 (Bohme et al.).** Let G = (V, E) be a plane graph. Let C = v_0v_1 ... v_kv_0, k ≤ 6, be the cycle that corresponds to the boundary of a face of G. If |L(v)| = 3 for all v \in P and |L(v)| = 5 for all v \in V - P, then G is not L-colorable.

Here we observe that if one of the exceptional cases of Theorem 3.13 occurs, it is the only such exceptional case.

**Lemma 1.14.** Let G = (V, E) be a planar graph and C = x_0 ... x_n, n ≥ 6, be the cycle that corresponds to the boundary of a face of G. Then G contains at most one bad vertex, bad edge or bad triangle.

**Proof.** First note that |V(C)| \geq 5 \cup 6, otherwise G does not contain any exceptional vertices. Note that if S is a vertex set of a bad edge or bad triangle, then V(C) \subseteq N(S) and |V(C)| = 6.

**Claim A.** If G contains a bad vertex u, then it cannot contain any other exceptional vertices. Assume an additional exceptional vertex v exists. Consider the set of edges of G corresponding to C and the edges between u and V(C). Deleting these edges from the plane creates five or six bounded regions and one unbounded region. Each of these bounded regions contains at most three vertices of V(C) on its boundary. The vertex v must be in one of these bounded regions. Since v is an exceptional vertex, it must have at least three neighbors in V(C). Since there are at most three vertices of V(C) in that region’s boundary, v has exactly three neighbors in V(C). Thus, v is part of a bad triangle (u, v, v') and u is...
adjacent to five vertices of V (C). The vertex v from that bad triangle must lie in the same region as v. The vertices v and v are each adjacent to three vertices of V (C) and have one common neighbor in V (C). So together, they must be adjacent to a total of five vertices of V (C), a contradiction. See Figure 3.2a for verification. Thus, given a bad vertex, G cannot contain any additional exceptional vertices.

**Figure 3.2 Exceptional vertex u**

**Lemma 3.14**

Claim B. If G contains a bad edge u_i u_j, then the only exceptional vertices of G are u_i and u_j.

Assume an additional exceptional vertex v exists. Consider the set of edges of G corresponding to C and the edges between u_i, u_j and F. Deleting these edges from the plane creates regions each with at most two vertices of V (C) on their boundary. Since v is an exceptional vertex, it must have at least three neighbors in V (C). Therefore, there does not exist a region in which v could lie without contradicting the planarity of G.

Claim C. If G contains a bad triangle (u_i, u_j, v_k), then the only exceptional vertices of G are u_i, u_j and u_k.

The proof of this statement is similar to that of Claim B and left to the reader. The lemma follows by the three claims above.

The following proposition is almost identical to Theorem 5.3 of [57], with the added condition that H contains all precolored vertices. The proof is included for completeness.

**Proposition 1.15.** Let G be a planar graph and P a set of vertices. Let L be an assignment of lists of colors such that |L(v)| = 1 for v ∈ P and |L(v)| = 5 for v ∈ V (G) - P. If there is an induced connected subgraph H of G containing all vertices from P such that it can be nicely colored with respect to L, then G is L-colorable.

Note if d(v, H) ≤ 2 for each v ∈ V (H) then every proper coloring of H is a nice coloring.

**Proof.** Consider a nice coloring c of H. Then |L_e| ≥ 3 for all e ∈ E (H) and |L_e| = 5 for all e ∈ G - E (H). Therefore, by Thomassen’s 5-list-coloring theorem, G - E (H) is L_e-colorable. Together with the coloring c of H, this gives a proper L-coloring of G as L_c(v) ⊆ L(v) for all v ∈ G - E (H).

**Lemma 1.16.** Let S be a shortest (u,v)-path in a planar graph G, where S = v_0 v_1 ... v_m with u = v_0, v = v_m. Then the following properties hold:

1. For all i, w ∈ N (S), d(w, S) ≤ 3.
2. For every x, y ∈ V (S), x ∼ y in G unless \{|x, y| \} = \{|v_i, v_{i+1}| \}, for i = 0, ... , m - 1.
3. If d(w, S) = 3 for some w ∈ N (S), then w ∼ \{|v_i, v_{i+1}, v_{i+2}| \}, for i = 0, 1, ... , m - 2.
4. If there is no separating 3-cycle or 4-cycle in G, then for each i with i = 0, 1, ... , m - 2 there is at most one vertex w ∈ N (S) such that w ∼ \{|v_i, v_{i+1}, v_{i+2}| \}.

**Proof.** Items (1)-(3) hold because S is a shortest (u,v)-path. To see the validity of item (4), assume there are two vertices adjacent to v_i, v_{i+1}, v_{i+2}. Then it is easy to verify that there is either a separating 3-cycle or a separating 4-cycle in G.

Note that Lemma 3.16 implies that if S is a shortest path between two vertices of a planar graph G, then every block of Q(S) with at least three vertices of S will have the following vertex set:

\{v_i, v_{i+1}, v_{i+2}, v_k \}

**Figure 3.3 Block in Q(S) where the bold line indicates S**

for some k ≥ 2, where v_i, v_{i+k} are consecutive vertices of S and w_j ∈ \{|v_i, v_{i+1}, v_{i+2}| \}, for j = 1, ..., k - 1. Observe that because S is a shortest path and there are no separating 3-cycles or 4-cycles in G, the vertices of Q(S) - S form an independent set. We call a block of Q(S) with i vertices of S an i-block, i = 2, 3, 4, .... See Figure 3.3 for examples of blocks in Q(S).

Note also that the block-cut-vertex tree of Q(S) is a path. Note that if Q (S) has a cut-edge, that edge is in S, and if Q(S) has a cut-vertex, that vertex is in S. We shall need a notion of a nontrivial block which will allow us to focus on subpaths of S and not worry about the boundary conditions. For a shortest (u, v)-path T, we say an edge e is a nontrivial cut-edge of Q(T ) if e is a cut-edge not incident to either u or v; we say B is a nontrivial block of Q(T ) if B is a block that does not contain u or v. We say a block B is a remote nontrivial block of Q(T) if |V (B) ∩ V (S)| = |V (B) ∩ V (S)| = 1 where B_1 and B_2 are distinct nontrivial blocks of Q(T). Let u, v ∈ V (S) and let T = u v v. If e is a nontrivial cut-edge in Q(T), then it is easy to see that e is a cut-edge in Q(S ); if B is a nontrivial block of Q(T), then B is a block of Q(S ).

**CONCLUSIONS:**

We proved the question of Albertson has a positive answer if there are no short cycles separating precolored vertices and there is a nice tree containing precolored vertices. We note here that by the definition of a (P,d)-Steiner tree, Theorem 1.4 can be applied to plane graphs with precolored vertices that are not far apart. For example, let G be a 100-cycle with vertices v_0, v_1, ..., v_99 and P = \{|v_1, v_{30}, v_{61}| \}. Then G contains a (P, 48)-Steiner tree obtained from deleting v_0v_99 and incident edges. The centers of the branches are far apart, but dist(v_1, v_{98}) = 3.

**REFERENCES:**