A NOVEL APPLICATION OF DIFFERENCE EQUATION FOR FERN LEAF DEVELOPER AS AN INVERSE PROBLEM

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ABSTRACT

Various analytical methods have been proposed to specify the geometrical orientation of the standard brands descriptions of a fern leaf. These are studied from two dimensional geometrical transformations called Iterated Function Systems. Here we have arrived at a novel design structure of a fern leaf using the nonlinear and probabilistic behaviour of generated points on the edges and interior patterns of the fern leaf. We claim such method can be used for other types of geometrical designs of plant leaves and biological species, which are self-similar obeying axial symmetry in the designer. In this paper, we focus our attention on inverse problems that consist in recovering random number sequences, which are the solution of a second order difference equation. Thus we assume that the structure of the fern leaf, boundary and matching conditions in the vertices are known a priori. The collections of boundary points are taken from the fern leaf, which generates the randomized affine transformations. Mathematica computational package is used to generate sequences of random numbers for both left and right boundary curves of the fern leaf. The second order difference equation is used for arriving at the decision of our novel method.

KEYWORDS: fern leaf generation, inverse problem, second order difference equation, self-similar sets, random iterated function system.

1 INTRODUCTION

Benoit Mandelbrot refers to the word "fractals" as objects which possess self-similarity [2]. Lindenmayer systems (L-systems) were invented by Aristid Lindenmayer and this is a way to model biological growth [5]. Michael F. Barnsley has studied the Construction of fractal objects with Iterated and Random Iterated Function Systems [1]. Przemyslaw Prusinkiewicz and Mark Hammel have studied the generation of fern leaf using Language-Restricted Iterated Function Systems, Koch Constructions, and L-systems [6]. In all these methods, one can generate any geometrical figure, plant leaf structure and other biological structure pertaining to micro and macro species. In all these generation of fern leaf, we understand the basic building of generation of fern leaf is a self-similar set. For a fern leaf this self-similar set is composed of points undergoing specific translation and rotation. This we mathematically adopt as an affine transformation. In this paper, by proper analysis of sequence of points on the boundary, interior and key turning points of the fern leaf are entered in a graph. The detailed discussion led to a systematic method which will help us to generate the fern leaf which comes under computational biology.

2 PRELIMINARY RESULTS

In [11], we have studied the oscillatory and nonoscillatory properties of second order neutral delay difference equation of the form

\[ \Delta \left( p_0 x_{n+1} + p_1 x_{n+1} - q_{n-\sigma} x_{n-\alpha} \right) + r_2 f^{\nu}(x_{n-1}) = 0 \]  

where \( p_0 > 0, q_0 > 0, \tau > 0, \sigma \geq 0, 1, 0 \leq p_0, q_0 \leq 0, \) for \( n \in \mathbb{N} = \{0, 1, 2, \ldots\}, \tau < \alpha, \in \{\pm \alpha, \ldots, 0\}, s \geq a, v \) is the ratio of odd positive integers such that \( v < 1, \Delta \) is the forward difference operator defined by \( \Delta x_n = x_{n+1} - x_n \) and the continuous function \( f : \mathbb{R} \to \mathbb{R} \) is nondecreasing in \( u \) such that \( uf(u) > 0, \) for \( u \neq 0 \) and

- \( C1) \ 0 < p_0 \leq p < 1, \)
- \( C2) \ 0 < q_0 \leq q < 1, \)
- \( C3) \ p > q, \)
- \( C4) \ \lim_{n \to \infty} \sum_{k=0}^{n-1} r_k < \infty, \) for \( M > 0. \)

Lemma 2.1(See.[11]) In addition to the conditions

- \( C1), \ (C2), \ (C3), \) and \( (C4), \)
- \( \lim_{x \to \infty} \frac{f^{\nu}(u)}{|u|} > 0, \) for some \( a \geq 1. \) If there exists \( a \leq 1, \)
- \( \lambda > \frac{\log a}{\sigma - 1} \)
- \( \frac{\sum_{k=0}^{n-1} r_k}{\left(1 - \frac{1}{\lambda}\right)} \to \infty \)
- \( f^{\nu}(x) \to \infty, \) for all \( c > 0 \) hold. If
- \( \lambda \nexists \infty, \) then every solution of equation (1) is oscillatory.

We use the following notations throughout, \( N = \{0, 1, 2, \ldots\}, \) the set of natural numbers including zero; \( N(a) = \{a, a+1, a+2, \ldots\}, \) where \( a \in N. \)

By a solution of equation (1), we mean a real sequence \( \{x_n\} \) which is defined for all \( k \geq \tau \geq 0 \) and satisfies the equation (1) for sufficiently large \( k \in N, a \in N. \) The equation (1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and otherwise it is oscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

A coordinate transformation of the form \( x' = a_{ax} x + a_{ay} y + b_x, \ y' = a_{yx} x + a_{yy} y + b_y \) is called a two dimensional affine transformation. Each of the transformed coordinates \( x' \) and \( y' \) is a linear function of the original coordinates \( x \) and \( y, \) and parameters \( a_{ij} \) and \( b_i \) are constants determined by the transformation type. Affine transformations having the parallel lines and finite points map to finite points. Translation, rotation, scaling, reflection, and shear are examples of two-dimensional affine transformations. Any general two dimensional affine transformation can
always be expressed as a composition of these five transformations. Another affine transformation is the conversion of the coordinate descriptions from one reference system to another, which can be described as a combination of translation and rotation. An affine transformation involving only rotation, translation and reflection preserves angles and lengths, as well as parallel lines. For these three transformations, the lengths and angle between two lines remains the same after the transformation.

Random number generators[10] are commonly used in particle-code simulations and in Monte Carlo simulations. With the transformation of two uniform random variables \( x_1, x_2 \), \( 0 < x_1, x_2 < 1 \),
\[
y_1 = \sqrt{-2 \ln x_1} \cos 2\pi x_2, \quad y_2 = \sqrt{-2 \ln x_1} \sin 2\pi x_2,
\]
where \( y_1 \) and \( y_2 \) both are Gaussian random variables with \( m = 0 \) and \( \sigma = 1 \). This transformation has been implemented as gaussv() and is available for use. Gaussian with mean \( m \) and variance \( \sigma^2 \) is often written as \( N(m, \sigma^2) \).

Throughout, this section let (h2) 0 and (h3), assume that \( 0 < q < p < min(q, p) < 0 \) and the condition \( n \equiv q \mod q \).

This implies that \( x_n < 0 \) for large value of \( n \), which is a contradiction to \( x_0 > 0 \). Thus, \( \Delta z_n > 0 \) must be hold for \( n \in N(n_1) \). It follows that \( z_n \) is increasing sequence for all \( n \in N(n_1) \). Therefore there exist two possibilities: either \( z_n > 0 \) or \( z_n < 0 \) for \( n \in N(n_1) \).

Suppose that \( z_n > 0 \) holds for \( n \in N(n_1) \). Then from equation (3), we obtain
\[
z_{n+\sigma - 1} > -(q_{n+\sigma - 1} + q_{2n+\sigma - 1})(x_{n-1}) \geq -q(x_{n-1}),
\]
where \( q = q_1 = q_2 \).

By triangular inequality we obtain \( x_{n-1} \geq \frac{1}{Q} z_{n+\sigma - 1} \) for \( n \in N(n_1) \), where \( Q = q^2 \). It follows from the equation (1) that \( A x_n + r_n f(x_n) \left( \frac{1}{Q} \right) z_{n+\sigma - 1} \leq 0 \), for \( n \in N(n_1) \).

The rest of the proof is same as in that of lemma 3.1.1. Hence we omit it.

**Theorem 3.1.2** In addition to the conditions (h1), (h2) and (h3), \( \sigma > 0 \) and the condition
\[
0 < \int_0^\infty \frac{dx}{f(x)} \int_0^\infty \frac{dx}{f(x)} < \infty, \text{ for } \epsilon > 0 \text{ hold. If}
\]
\[
\sum_{s=0}^{\infty} r_s = \infty,
\]
then every solution of the equation (1) is oscillatory.

The proof of the theorem 3.1.2 is same as in that of theorem 2.2. Hence reader has left the proof.

**3.1.3 Example**

We give the following example to illustrate the results.

Consider the difference equation
\[
A(x_n + r_n f(x_n)) + \frac{\sqrt{2}}{3(2K + (n + 3)(n + 11)^{1/2})} x_{n+1} \leq 0,
\]
for \( n \in N(1) \), (5)
where \( p_n = \frac{1}{3} \), \( q_n = \frac{3}{4} \), \( q_{2n} = \frac{3}{4} \), \( \sigma_1 = 1 \), \( \sigma_2 = 2 \), \( \nu = \frac{1}{3} \), \( r_n = \frac{\sqrt{2}}{3(2K + (n + 3)(n + 11)^{1/2})} \).

All the conditions of theorem 3.2 are satisfied. Hence all solutions of equation (5) are oscillatory.

In fact, \( \{ x_n \} = \left\{ K + \frac{1}{2}(n^2 - 8n) \right\} \) is one such solution of equation (5), where
Table 1. (List of points and the corresponding affine transformations obtained from the image of a fern leaf (Fig.1))

<table>
<thead>
<tr>
<th>S.No.</th>
<th>POINTS FROM THE BOUNDARY OF FERN LEAF</th>
<th>CORRESPONDING AFFINE TRANSFORMATION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X, y</td>
<td>x</td>
</tr>
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<td>25.315</td>
</tr>
<tr>
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<td>17.2</td>
<td>25.31</td>
</tr>
<tr>
<td>10</td>
<td>17.1</td>
<td>25.3</td>
</tr>
<tr>
<td>11</td>
<td>17</td>
<td>25.25</td>
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<td>25.19</td>
</tr>
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<td>14</td>
<td>16.3</td>
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<tr>
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<td>25.1</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>25.05</td>
</tr>
</tbody>
</table>

In this paper, we have identified a method which is computationally efficient and which generalizes all existing methods for self-similar collection of basic building materials of plant objects. The following steps illustrate the materials needed and the methods used:

1. We have chosen an image of a fern leaf (Fig. 1).
2. We have listed points on the boundary (both LHS and RHS) (Fig. 2) of the fern leaf (Table 1).
3. We constructed the affine transformations for the points listed in Table 1.
4. We obtained the frequency distribution of the affine transformation (Table 2).
5. By fitting the boundary of the fern leaf to the normal equation $y = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ (Fig. 3, Fig. 4), we obtain the mean and standard deviation of the fitting curves (Table 3).
6. We generated the sequence of random numbers using matlab coding in normal distribution.

Matlab Programming for random number generation (LHS):

```matlab
for i = 1:632
    y = random('Normal', 9.7434, 0.0146);
    r = y/25;
    disp(y)
end
```

Matlab Programming for random number generation (RHS):

```matlab
for i = 1:163
    y = random('Normal', 16.6391, 0.1416);
    s = y/32;
    disp(s)
end
```

7. We consider the second order nonlinear neutral delay difference equation

$$\Delta^2\left(x_n + p_x x_{n-1} - q_n x_{n-1} - q_{2n} x_{n-2}\right) + r_n f_n(x_{n-1}) = 0,$$

where $p_n > 0$, $q_n > 0$, $p_1 > 0$, $\sigma_1 \geq 0$, $q_{2n} \leq 0$, for $i=1,2$ and $n \in \{0, 1, 2, ...,\}$, $\tau < \min(\sigma_1, \tau_1) \in \{-s, ..., 0\}$, and $s \geq \max(q_{2n})$.

8. By choosing $p_n = \frac{1}{3}$, $q_n = \frac{3}{4}$, $q_{2n} = \frac{3}{4}$, $\tau = 1$, $\sigma_1 = 2$, $\sigma_2 = 3$, $\nu = \frac{1}{5}$, we have the difference equation

$$r_n = \sqrt{\frac{k}{3(2K + (a + 3)(n + 1))}},$$

$$f_n(x_{n-1}) = x_{n-1}^{\tau} + \frac{3}{4} x_{n-1}^{3/4} - \frac{3}{4} x_{n-1}^{3/4} - \frac{3}{4} x_{n-1}^{3/4} = 0.$$
In the LHS, we have collected 633 points (Table 1). Rest of the points are midpoints of the existing points.

<table>
<thead>
<tr>
<th>S.no.</th>
<th>Frequency</th>
<th>Cumulative frequency</th>
<th>H</th>
<th>Frequency</th>
<th>Cumulative frequency</th>
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<td>4</td>
<td>1</td>
<td>-1</td>
</tr>
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<td>12</td>
<td>3</td>
<td>2</td>
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<td>3</td>
<td>3</td>
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<td>4</td>
<td>4</td>
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<td>6</td>
</tr>
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<td>7</td>
</tr>
<tr>
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<td>136</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
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<td>24</td>
<td>160</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
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<td>0.0225</td>
<td>88</td>
<td>248</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 3. (Mean and Standard deviation for the frequency distribution)

<table>
<thead>
<tr>
<th></th>
<th>LHS</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>µ (Mean)</td>
<td>9.7434</td>
<td>16.6391</td>
</tr>
<tr>
<td>σ (Standard Deviation)</td>
<td>0.0146</td>
<td>0.1416</td>
</tr>
</tbody>
</table>

Fig. 1 Image of a fern leaf

Fig. 2 Boundary of the image of the fern leaf

Fig. 3 Frequency curve for LHS of boundary of fern leaf

Fig. 4 Frequency curve for RHS of boundary of fern leaf

Fig. 5 Boundary of the fern leaf obtained from affine transformations
4 RESULTS AND DISCUSSION
The plotted points taken from Table 1 provide us a collection of affine transformations, which give a probability distribution for that behaviour. We attach a random number sequence born out of an oscillatory sequence which is a solution of a specific difference equation. The output of the fern leaf from LHS and RHS are computationally generated and plotted in the Fig. 2. After completion of LHS, RHS and centre line of the fern leaf can be obtained by zooming down for self-similar geometrical structure which is posted at stipulated points on the centre line to obtain final output as fern leaf. Corresponding figures and tables are presented to support our claim. In future, we include the method for a larger collection of points which will produce more number of accurate affine transformations, generating very close approximate points in the construction, making the design more appropriate with additional colour specification.

REFERENCES
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